Through the Looking Glass: Inverse Trigonometric Functions

UNH Mathematics Center

In this section we will "turn the trig functions inside out and read them backwards." Much of what we say about the inverse trigonometric functions can be said of *any* inverse functions at all, so that is where we'll begin.

The inverse trigonometric functions, judging just by their graphs, are so odd-looking, that you might wonder just what they are good for.

It's a fair question. The inverse trig functions, especially the inverse sine, inverse tangent and inverse secant functions, turn out to be very handy as *antiderivatives* for a variety of integrals involving quotients and roots of polynomials.

The reasons for this surprising development, in a nutshell:

- Polynomials with real coefficients can all be rewritten as products of linear, and irreducible quadratic, factors.
- The trigonometric functions satisfy the "Pythagorean identity" $\sin^2(x) + \cos^2(x) = 1$.

1 The sine function and its inverse

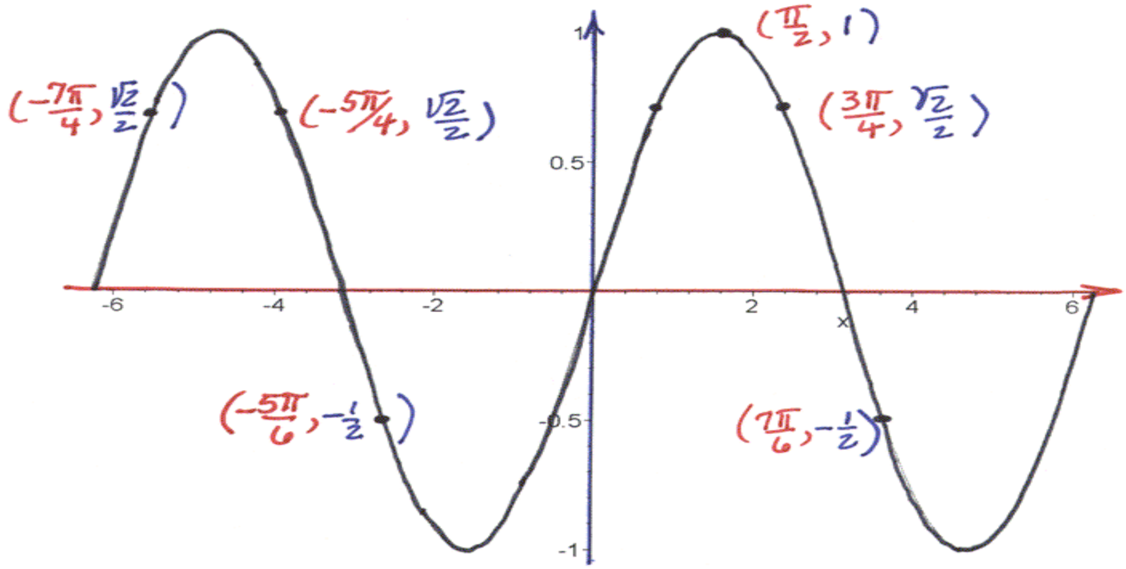
The *sine function*, as you will recall, consists of a whole lot of ordered pairs of real numbers. The following ordered pairs all belong to the sine function, as do endless others:

$$(0,0),\ \left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right),\ \left(\frac{-4\pi}{3},\frac{\sqrt{3}}{2}\right)$$

If you assembled all these ordered pairs, interpreted each pair as the coordinates of a point, and plotted the points, you would obtain the graph of the sine function.

1.1 The sine function is periodic, not one-to-one.

Here is a picture of (part of) the sine function's graph.



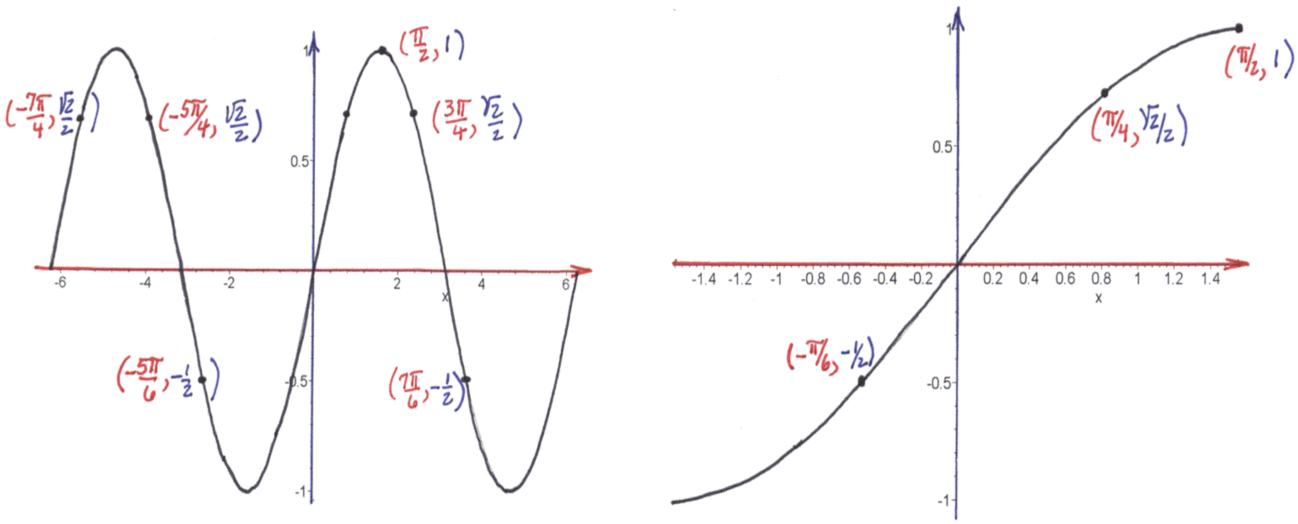
We have color-coded this sketch: the horizontal axis shows up on your screen, we hope, in red, and the vertical axis in blue. Each marked point's first coordinate is labeled in red and its second in blue.

The graph shows two periods of the sine function, with input numbers between -2π and 2π. The sine function's *whole* domain would be the entire set of real numbers; its range is the interval $[-1,1]$. The sine function is *periodic*: its graph is wonderfully repetitive. We've included four points whose second coordinate is $\frac{\sqrt{2}}{2}$ (**A question for you:** what is the first coordinate of the *unmarked* point on the graph?)

But the sine function's periodicity means that we cannot use the entire function when we construct its inverse; the *inverse sine function*, or the *arcsine function*. A function must always return *a specific value* for each input number: **If *x* is a number and *f* is a function, then *f(x)* must be the unambiguous *name of one specific number*.**

1.2 Selecting a one-to-one portion of the sine function

The first thing we do is to select a portion of the sine function that is one-to-one (each input number is associated with exactly one output number, and *vice versa*). There would be lots of ways this could be done, and the graph below shows an advantageous way to make the choice.



The sketch to the left is the sine function's graph. The sketch to the right is a portion of the sine function that represents a one-to-one function.

Notice that the *values* of the restricted sine function (the graph on the right) still range from -1 to 1, so that in a way we have kept "as much of" the function as we could, while making our graph show a *one-to-one* function instead of a periodic one.

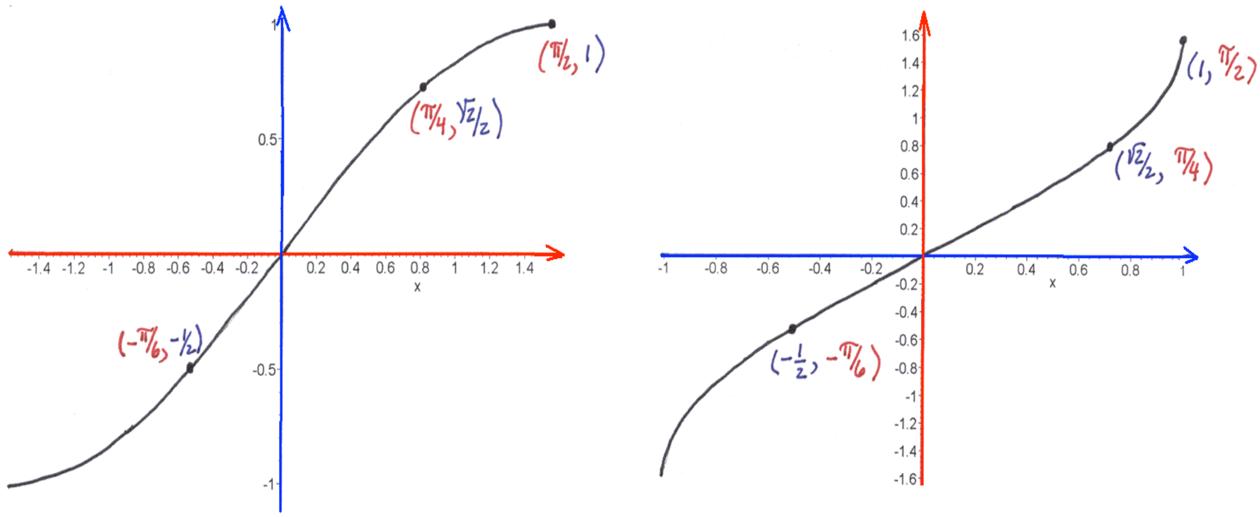
There would have been *many* ways to choose just one section of the sine function's graph, in "selecting" a piece of the sine function to be one-to-one. Yes, there is a reason for making this particular choice: No, we are not going to burden you with it just now.

1.3 Making an inverse for the sine function

We make an inverse function out of original function's ordered pairs, by reversing each pair. The nice thing about a one-to-one function is that after we reverse the function's ordered pairs, the reverse pairs still satisfy the definition of "function."

> A function is a collection of ordered pairs of numbers, **such that no two different ordered pairs have the *same first element***.

Functions are allowed to have many different pairs with the same *second* element ... and periodic functions certainly do! It's when the ordered pairs are then *reversed* to make the inverse function, that we need to start with a one-to-one function.



The sketch to the left is the restricted sine function's graph. The sketch to the right is the inverse sine function, or arcsine function.

A really good long look at the two graphs above will show you the whole business in a nutshell.

- The first graph shows the "restricted" one-to-one portion of the sine function. We've marked the horizontal axis (where the input numbers live) in red. We've labeled three sample graph points, first coordinates in red. The output numbers (second coordinates), and their (vertical) axis, are in blue.
- On the right-hand graph you can see that **we've made an inverse function by reversing the original function's ordered pairs.** We've kept the same colors, so that now the *input numbers* and the horizontal axis are in blue. The *output numbers* and the vertical axis are in red.
- Compare the two graphs! Every single point on one graph has had its coordinates reversed before being re-plotted on the other graph.
- One graph can easily be turned into the other. *If we just got the red and blue axes in either picture repositioned, as they should be in the other picture, all the rest of the graph would follow!*
- You might think you could reposition the axes by rotating the sketch counterclockwise through one right angle, but it doesn't work that way. If you did that, the "previously vertical" axis would be horizontal, but the function would be pointing in the wrong direction.
- It will help you figure this out if you do it on paper. You can reposition the axes (and with them, the rest of the graph) this way:
    - Sketch the graph on paper. Pick up the paper, and turn it over. Hold it up to the light, and *look at it from the back!*
    - Now rotate the paper until you can see the "old" horizontal axis pointing upwards. The "old" vertical axis will now be horizontal, and its increasing direction will point to the right.
    - That's it! What can now see the same ordered pairs you had before, *but their coordinates are reversed*. You are looking at a graph of your function's inverse.
- The domains and ranges switch places. When we make an inverse for a one-to-one function, all the first coordinates of points from the original function become *second*

coordinates of points on the inverse function's graph ... and vice-versa. That is, the original function's domain becomes the inverse function's range; and the original function's range becomes the inverse function's domain.

The function we end with is called the "inverse sine" function. It's also known as the "arcsine" function, and often enough it's called "$\sin^{-1}$."

<u>WARNING:</u> the inverse of a function (represented by a -1 as an exponent) *DOES NOT* mean the reciprocal of the function. In other words,

$$\boxed{\sin^{-1}(x) \neq \frac{1}{\sin(x)}}$$

Remember that the inverse of a function is found by reversing the original function's domain and range values. The domain of the inverse sine function is the interval of real numbers $[-1,1]$. For any number x in that interval, arcsin(x), also known as $\sin^{-1}(x)$, is **the number between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, whose sine value is *x***. In other words, the range of the inverse sine function is the interval of real numbers $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. For instance:

- Many numbers have sine values of $\frac{\sqrt{2}}{2}$. Some examples are:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{-5\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

However, there is only one arcsine value of $\frac{\sqrt{2}}{2}$:

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

- Many numbers have sine values of -1. Some examples are:

$$\sin\left(\frac{3\pi}{2}\right) = -1,\ \sin\left(\frac{-\pi}{2}\right) = -1,\ \sin\left(\frac{7\pi}{2}\right) = -1$$

However, there is only one arcsine value of -1:

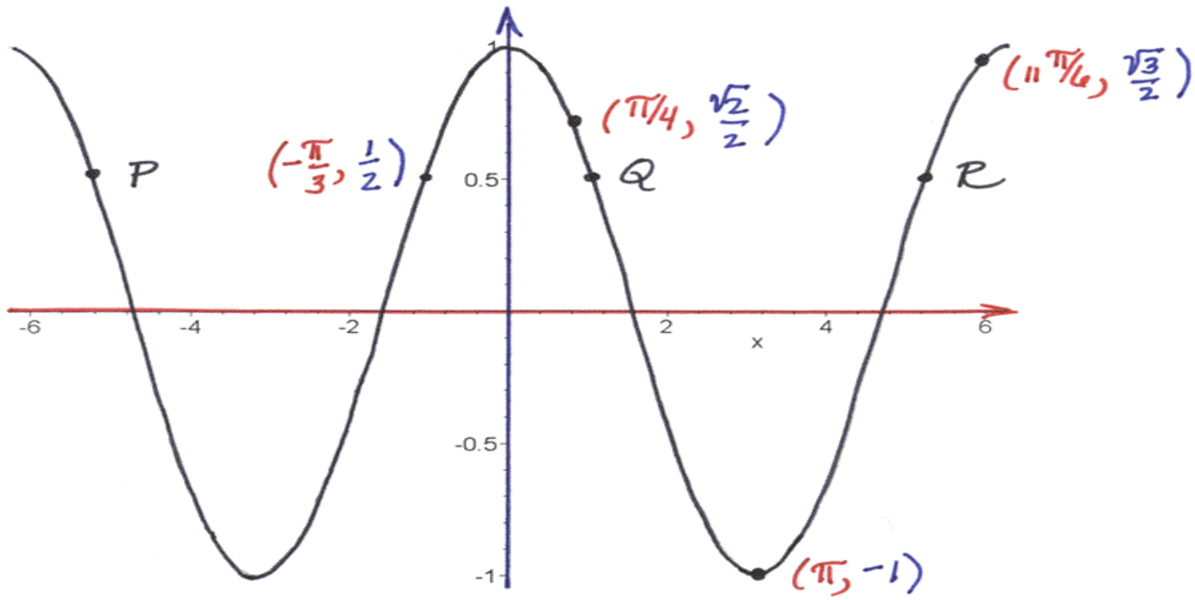
$$\sin^{-1}(-1) = \frac{-\pi}{2}$$

2 An inverse for the cosine function

The cosine function can have an inverse too. The function *arccosine* is not used as much as the inverse sine function, because (as it will turn out) the uses in calculus we might have for an inverse cosine function can all be done just as well with the inverse sine. But, that makes it good to practice on!

2.1 The cosine function is periodic, not one-to-one

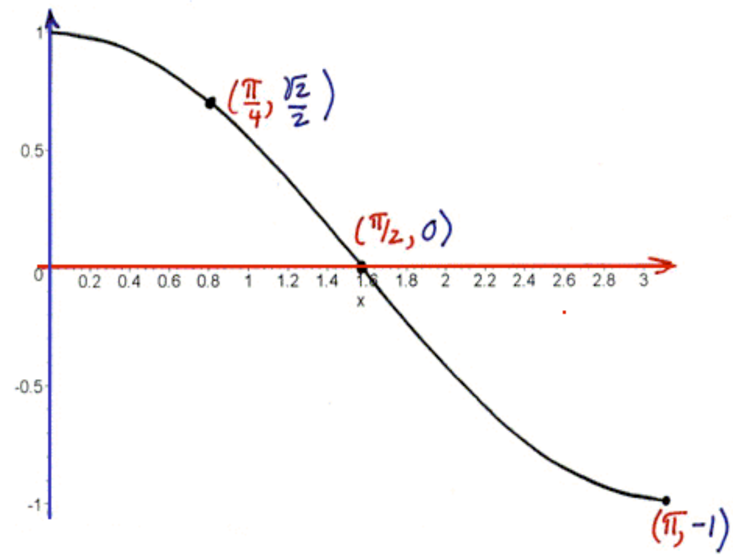
And so, just as with the sine function, it has no inverse function as it stands. Below is a sketch of two periods of the cosine function's graph.



There are an infinite number of example sets of points on the graph, with identical second coordinates. *What are the coordinates of the graph points labeled P, Q and R? Notice that these three points all have the same second coordinate.*

2.2 Selecting a one-to-one portion of the cosine function

As before, we need to *restrict* the cosine function (so that the portion we "keep" is one-to-one) in order to form an inverse function from it. We choose to "keep" the portion of the graph for input numbers between 0 and π:



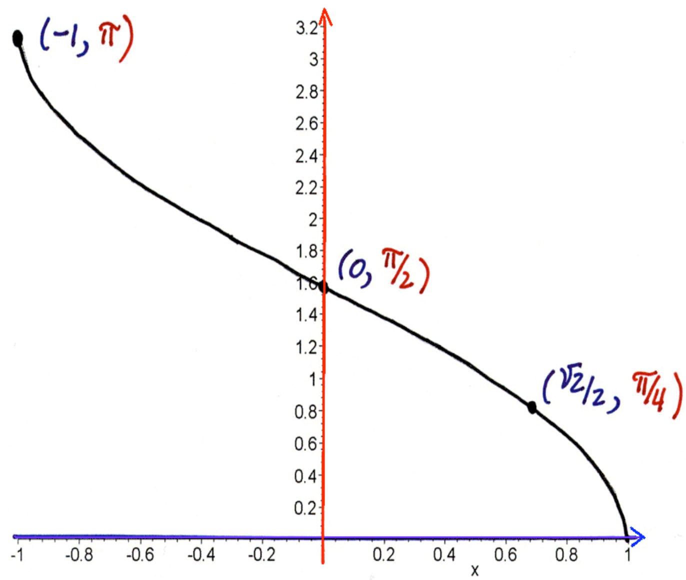
Our abbreviated graph represents a *restricted* cosine function. It's only a little piece of its former self, but it now has the advantage that it's a one-to-one function. You can see that it passes the "horizontal line test" ... No horizontal line meets this graph more than once.

2.3 Making an inverse for the cosine function

Now it's your turn. Begin a sketch of the inverse cosine's graph on scratch paper now ... **DON'T LOOK AT THE NEXT PAGE YET!** ...

- First, provide new horizontal and vertical axes. Copy onto them (approximately) the same units as are on the restricted cosine graph's *vertical and horizontal* axes, respectively.
- Then, put a few points on your graph. You can use the marked points on the restricted cosine graph as landmarks. For each point, **reverse its coordinates,** and plot the reversed-coordinates-point on your graph.
- The *domain* of your inverse function should be the range of the cosine function, which is the set of numbers $[-1, 1]$.
- The *range* of your inverse function should be the domain of the *restricted cosine function*. What is that? (Hint: it shouldn't have any negative numbers in it.)
- Now you can "fill in" the rest of your inverse function's graph. Make up more landmark points if you like, or copy the restricted cosine function's graph onto paper, turn it over, and hold it up to the light! (If you prefer, you can imagine rotating it around the line $y = x$; it amounts to the same thing.)

Ready? You should have a picture like the graph below.



This is a graph of the *arccos* or *inverse cosine* function, often written with the −1 exponent as $\cos^{-1}$. Its domain is [−1,1], the same as the range of the cosine function. Its *range* is [0,π], which is **part of the cosine function's domain**. (That is, *all* of the *restricted cosine function's domain*.)

For example, $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ because

- $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ **and**
- $\frac{\pi}{4}$ is between 0 and π, thus $\frac{\pi}{4}$ is in the range of the arccosine function.

3 The inverse tangent function

We'll just describe quickly how the inverse tangent function is constructed. After following the description of the inverse sine and cosine function, we think you can follow the construction of the inverse tangent easily. It's probably in your textbook just waiting for you. Just a few outline points:

- The tangent function is periodic, with period π. It has vertical asymptotes π units apart, and its range is the entire set of real numbers.
- The "selected portion" of the tangent function's graph, for the purposes of constructing an inverse function, is the portion between the vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$. If you delete all the rest of the tangent graph, you can see that this remaining section of it is a one-to-one function.

- You might select landmark points $\left(\frac{-\pi}{3},-\sqrt{3}\right)$, $\left(\frac{-\pi}{4},-1\right)$, $(0,0)$, $\left(\frac{\pi}{4},1\right)$, $\left(\frac{\pi}{3},\sqrt{3}\right)$.
- When you reverse all these points remember that the asymptotes reverse too. Everything vertical becomes horizontal! Your inverse function will have *horizontal* asymptotes.
- The inverse tangent function's domain is the whole set of real numbers. The range is an *interval* of real numbers. (Does the range-interval include its end points, or not? Think about those horizontal asymptotes!)
- You can find a picture of the inverse tangent function's graph ("arctan" or "$\tan^{-1}$") in your text. Probably on your calculator too! Does it look like your sketch?
- What (approximately) is arctan(75,000)? Where is it on your graph? What (approximately) is arctan(−0.02)? Is it positive or negative? Where is it on your graph? Can you give (without looking it up) an approximate value for arctan(−1.15)?