The Power in Numbers: A Logarithms Refresher

UNH Mathematics Center

For many students, the topic of logarithms is hard to swallow for two big reasons:

- Logarithms are defined in such a "backwards" way that sometimes students see only the rules for using them, and can't get a good picture of what they *are*.

- Tables of logarithms were originally constructed to make calculations easier. And of course everybody uses calculators for this purpose now! So it often happens that students find themselves in a university calculus course without much prior experience with logarithms.

It's not hard to see why the definition of a logarithm has this backwards-looking flavor: the log functions are, after all, *inverses* to important exponential functions. But this is the very reason they are still indispensable!

They are the functions we need, (in combination with their companion exponential functions) to describe exponential growth.

Every situation in which the growth-rate of a quantity is proportional to its present level is described by an exponential function.

Example: Although a country's birth rate is affected by other factors as well, it will be proportional to *the country's present population*. Every year, there are more babies born in New York City than in Durham, New Hampshire. This is true of other kinds of "populations" as well: dollars, or fish, or bugs, or radioactivity. The logarithm and exponential functions describe them all.

1. Why use two systems of logarithms?

When the first tables of logarithms were worked out (to help 17th-century sailors do the calculations that kept them from being lost on the seas) they were based on a decimal number system. Count your fingers and you'll see why.

The really odd thing is, that nature loves logarithms too (and Mother Nature isn't biased in favor of the number 10). If you take calculus you will see that the definite integral

$$\int_1^x \frac{1}{t}\,dt$$

(which is a function of its variable upper limit x) has all the properties a logarithm function ought to have. It is the logarithm function that Nature provides. Its corresponding exponential function is exp(x), or e^x; and its base is the "natural number" e ≅ 2.718.

The 10-based "common" logarithms and the natural logarithms follow the same rules. Their values are even proportional to each other.

It would actually be possible to construct a system of logarithms based on *any* positive number. Most people figure, the two we have will do nicely.

2. Defining logarithms

We'll start with the common, 10-based logarithms. The idea is to think of each number as a power of 10:

Number	**Re-written as a power**	**Number's logarithm**
1	10^0	0
10	10^1	1
100	10^2	2
1000	10^3	3
10000	10^4	4
1,000,000	10^6	6
0.01	10^{-2}	–2
0.00001	10^{-5}	–5

When a number is rewritten as a power of 10,
its common logarithm is just the exponent.

Of course, we don't have many numbers in our table yet! But you can already see a few things:

- When we multiply numbers, the logs are added. Notice, for example, that

$$\begin{aligned} \log(100\times 10{,}000) &= \log(1{,}000{,}000) \\ &= \log\,(10^6) \\ &= 6 \end{aligned}$$

  and also that

$$\begin{aligned} \log(100) + \log(10{,}000) &= \log(10^2) + \log(10^4) \\ &= 2 + 4 \\ &= 6 \end{aligned}$$

  Thus,

$$\log(100\times 10{,}000) = \log(100) + \log(10{,}000)$$

- When we divide numbers, the logs are subtracted. For example,

$$\log\left(\frac{100}{0.01}\right) = \log(10{,}000)$$
$$= \log(10^4)$$
$$= 4$$

and also that

$$\log(100) - \log(0.01) = \log(10^2) - \log(10^{-2})$$
$$= 2 - (-2)$$
$$= 4$$

Thus,

$$\log\left(\frac{100}{0.01}\right) = \log(100) - \log(0.01)$$

- If we square a number, its log doubles. For instance, the log of $(10^3)^2$ is 6, which is twice the log of 10^3.

- We only have logs for *positive numbers*. Some of the logs are negative: they are the logs of numbers smaller than 1.

The first tables of logarithms were constructed by John Napier, working in his castle at Merchiston in Scotland. It took him a good 20 years, and it's all the more remarkable because he didn't even have exponential notation to work with.

Nevertheless Napier's "wonderful reckoning numbers" are exponents. There are other ways of calculating logarithms now, and by the time you study power series in calculus you'll see how it can be done. For the time being, try out a few of them with your calculator. **You should be aware the logarithms are *real* numbers, which we can only *approximate* using decimals. So it's only to be expected that the decimal numbers you'll see require some rounding before you recognize them as the numbers they really are!**

- The common logarithm of 3 is (to five figures, at least) 0.47712. Try it on your calculator. Ask the calculator for $10^{0.47712}$, and your answer should be about 3.

- Use your calculator to approximate log(8). To five figures, it should be 0.90309. That tells us that $10^{0.90309} \cong 8$ (use your calculator to verify this).

- The sum of 0.90309 and 0.47712 is 1.38021. What number has 1.38021 as its logarithm? A number between 10^1 and 10^2, we think - because its logarithm is between 1 and 2. Ask your calculator for $10^{1.38021}$, and see if you don't get 24 = 8·3. When we add logs of two numbers, we get the log of their product.

- And while we're at it, ask your calculator for the common logarithm of 2.4. You should get (within rounding!) 0.38021, the fractional part of log(24).

- The common logarithms go well with decimal numbers. If we know that log 3.75 = 0.57403 then we also know that

    - $\log 0.375 = \log(3.75\times10^{-1}) = 0.57403-1$
    - $\log 37.5 = \log(3.75\times10^{1}) = 1.57403$
    - $\log 37{,}500 = \log(3.75\times10^{4}) = 4.57403$.

  The fractional part of the logarithm, 0.57403, gives us the significant digits 3.75 and its integer part tells us the power of 10 that multiplies the 3.75.

- If we divide 3 by 8, we get 0.375. So the logarithm of 3/8 should be about 0.47712 – 0.90309 = –0.42597. Although a calculator doesn't do it, it's been customary to *write* this number as 0.57403–1, because 0.57403 is the log of 3.75.

- There's no good way to find log(11) from log(8) and log(3). It's not their sum!

- One-third of 0.90309 (log 8) is 0.30103. Ask your calculator for $10^{0.30103}$. It should return 2, which is the *cube root* of 8. When we divide a number's logarithm by 3, we have the logarithm of the number's cube root.

2.1 A note about notation

The logarithm function is *sometimes but not always* written with parentheses around its argument (the number the function acts on). It's always OK to enclose the argument to a log function in parentheses.

One time you *must* use parentheses is if the argument to a logarithm function is a sum. If you want the logarithm of the number 2x+5, you *must write* "log(2x+5)". If you omit the parentheses and write "log 2x+5" everyone who reads your work will think you meant "(log 2x) + 5" or " 5 + log 2x." Unfortunately, this will include the person who reads your quiz papers!

2.2 The calculus-based logarithmic function

The logarithmic function that arises naturally out of calculus is called "ln." Its name is pronounced "natural log" or sometimes just "log" or by sounding out its spelling: "el-en."

Its inverse exponential function is "exp" and its value-variable is called either "exp(x)" or "e^x." You can pronounce it either as "exp" or "e-to-the-x." The first notation, exp(x), reminds us that the number x is an argument to the function: the second notation, e^x, reminds us of the rules of exponents that the function's values follow.

The number e itself, which is exp(1) or e^1, is an irrational number: 2.718 is only an approximation to its value. It is fine to use the name e instead of the approximate decimal value: e is, in fact, the correct name for this particular number.

Leonhard Euler (pronounced "oiler") was the key figure in 18th century mathematics, and you will guess correctly that the numerical base of the natural logarithms is still called e in his honor.

All the "rules" of logarithms are the same, whether we use 10 as a base (common logarithms) or e as a base (natural logarithms). The systems are even proportional!

3. Here are the Rules

$\log(xy) = \log(x) + \log(y)$	$\ln(xy) = \ln(x) + \ln(y)$
$\log(x/y) = \log(x) - \log(y)$	$\ln(x/y) = \ln(x) - \ln(y)$
$\log(x^y) = y\cdot\log(x)$	$\ln(x^y) = y\cdot\ln(x)$
$10^x\cdot 10^y = 10^{x+y}$	$e^x\cdot e^y = e^{x+y}$
$10^x/10^y = 10^{x-y}$	$e^x/e^y = e^{x-y}$
$(10^x)^y = 10^{xy}$	$(e^x)^y = e^{xy}$
$\log(1) = \log(10^0) = 0$	$\ln(1) = \ln(e^0) = 0$
$10^{\log x} = x = \log(10^x)$	$e^{\ln x} = x = \ln(e^x)$

4. Using the logarithm and exponential functions

The key to using these functions is to remember how good they are at undoing each other:

$$10^{\log x} = x = \log(10^x) \qquad e^{\ln x} = x = \ln(e^x)$$

which is to say, applying first one and then the other (in either order) to a number returns the original number!

Example 1: The equation

$$y = 6\ln(x) + 1$$

gives y "in terms of x" - meaning that y is written as an expression whose only variable is x.

Suppose we want to revise it, so that x is given in terms of y. (That is, we want to solve the equation for x.) We begin by solving it for ln(x):

$$\ln(x) = \frac{y-1}{6}$$

Now we apply the natural exponential function (we choose the natural exponential function instead of the tens-power one, because it's the companion function to the natural logarithm):

$$e^{\ln(x)} = e^{\frac{y-1}{6}}$$

We know that

$$e^{\ln(x)} = x$$

Therefore,

$$x = e^{\frac{y-1}{6}}$$

Example 2: Using a log function is the way to get "at" a variable that's part of an exponent:

Let's say we need to find the value $y = 50(10)^{2x}$ of x when y = 450.

We would first solve for the exponential expression:

$$(10)^{2x} = \frac{y}{50}$$

$$(10)^{2x} = \frac{450}{50}$$

$$(10)^{2x} = 9$$

Now we would use a logarithm function to undo the effect of the tens-power exponential function. Either the common or the natural log function will work, but here it seems in the spirit of things to use the common logarithm:

$$\log(10)^{2x} = \log(9)$$

We know that

$$\log(10)^{2x} = 2x$$

Thus,

$$2x = \log(9)$$

So, using the properties of logarithms,

$$2x = \log(3^2)$$
$$2x = 2\log(3)$$
$$x = \log(3)$$
$$x = 0.47712$$

Example 3: Suppose that the quantity *q(t)* of some substance depends on the time, *t*, and that the dependence is observed to be

$$q(t) = 400e^{0.15t}$$

with the time measured in hours. This is a very typical "model" of exponential growth, so it's good to be familiar with it.

- At "time zero" (when the discussion begins) the quantity of the substance was $q(0) = 400 \cdot e^0 = 400$. We apparently *started* with 400 units of the substance.

- Because the argument $0.15t$ to the exponential function is positive (usually in this model we think of time as going *forward*, so that $t \geq 0$) the quantity *q(t)* will *increase*. The positive exponent-coefficient 0.15 tells us that this an exponential model describes growth.

- An exponential growth model is often described by its time of doubling. How long will it take in this instance? We need to know the time t for which *q(t)* = 800:

$$800 = q(t) = 400e^{0.15t}$$

so that

$$e^{0.15t} = 2$$

To solve $e^{0.15t} = 2$, we would apply a log function. The natural log function seems most appropriate:

$$\ln(e^{0.15t}) = \ln(2)$$

$$0.15t = \ln(2)$$

We look up the log's value: $\ln(2) \cong 0.693$. Then,

$$t = \frac{\ln(2)}{0.15} = \frac{0.693}{0.15} = 4.62$$

The substance will have doubled in approximately 4.62 hours (about 4 hours 37 minutes).

Remember what we said earlier about approximations: although logarithms are *real* numbers we are using *decimal* numbers to approximate them. So we may be a minute or two off here. What's a minute among friends, after all?

The doubling continues about every 4.62 hours. At the end of about 9.24 hours (about 9 hours 22 minutes) we will have *four times* the quantity we began with.

5. Some Problems

Working the problems is always the best way to learn mathematics!

1. Below is an abbreviated table of common logarithm values.

Common Log	**Approximate Value**
log(2)	0.30103
log(3)	0.47712
log(5)	0.69897
log(7)	0.84510
log(9)	0.95424
log(11)	1.04139

Use the table above to evaluate the following. Note that some answers will have a "log(x)" term in it.

$\log(60)$

$\log(2/55)$

$\log(3500)$

$\log(5.5)$

$\log(77x^2)$

$\log\left(\frac{\sqrt{x}}{6.3}\right)$

2. Use the properties of the logarithm function to break apart the expression into its simplest components. Hint: that includes factoring 504, which is divisible by 8.

$$\ln\left[\frac{504x^2}{(y+1)^3}\right]$$

3. In our example of exponential growth, someone in some lab somewhere must have observed the substance carefully enough to come up with the model

$q(t) = 400e^{0.15t}$. How do you suppose it was done?

- It was easy to weigh the substance at the start of the experiment. That's where the 400 came from.
- The question remains: if we know $q(t) = 400e^{kt}$, how did the experimenters decide that k was approximately 0.15?

Apply the natural log function to $q(t) = 400e^{kt}$. You will get a linear expression for $\ln[q(t)]$. Explain how you could graph this linear expression to find k.

4. A model for exponential decay (the opposite of exponential growth) might be

$$q(t) = 650e^{-0.07t}$$

where $q(t)$ is measured in grams and t in hours. You can tell that this model describes decay rather than growth because of the negative coefficient –0.07 in the exponent. This is an example of a Radioactive Decay model.

   a. How much radioactive material was present at the start of the discussion?
   b. After 3 hours, how much material is still radioactive?
   c. What is the half-life of the substance?

5. We have put $2,500 in a bank account that pays 5% interest at the end of each year.
   a. What will the account balance be at the end of the first year? What is the balance, as a percent of the original $2,500? (It will surely be over 100%.)
   b. The years go by, and we have almost forgotten the bank account, although it is still earning 5% interest at the end of each year. Suppose that at the beginning of the 10th year the balance in this account is N dollars. Write an expression for its value at the end of the 10th year.
   c. How much is in the account at the end of ten years?
   d. It occurs to us that the bank balance is growing exponentially. Can we rewrite $2{,}500(1.05^t)$ in the "usual" exponential form

$$2{,}500e^{kt}$$

   for some number k?
   e. How long does it take for the money in this account to triple?

6. Some Answers

1. We're using the values

Common Log	Approximate Value
log(2)	0.30103
log(3)	0.47712
log(5)	0.69897
log(7)	0.84510
log(9)	0.95424
log(11)	1.04139

- Use both the rule for products and the rule for powers:

$$\begin{aligned}
\log(60) &= \log(2 \times 2 \times 3 \times 5) \\
\log(60) &= \log(2^2 \times 3 \times 5) \\
\log(60) &= \log(2^2) + \log(3) + \log(5) \\
\log(60) &= 2\log(2) + \log(3) + \log(5) \\
\log(60) &\approx 2(0.30103) + (0.47712) + (0.69897) \\
\log(60) &\approx 1.77815
\end{aligned}$$

- Use both the rule for quotients and the rule for products:

$$\begin{aligned}
\log\left(\frac{2}{55}\right) &= \log(2) - \log(55) \\
\log\left(\frac{2}{55}\right) &= \log(2) - \log(5 \times 11) \\
\log\left(\frac{2}{55}\right) &= \log(2) - \left[\log(5) + \log(11)\right] \\
\log\left(\frac{2}{55}\right) &\approx (0.30103) - \left[(0.69897) + (1.04139)\right] \\
\log\left(\frac{2}{55}\right) &\approx -1.43933
\end{aligned}$$

- Use the rule for products and the rule for $\log(10^x)$:

$$\log(3500) = \log(5 \times 7 \times 10 \times 10)$$
$$\log(3500) = \log(5 \times 7 \times 10^2)$$
$$\log(3500) = \log(5) + \log(7) + \log(10^2)$$
$$\log(3500) = \log(5) + \log(7) + 2$$
$$\log(3500) \approx (0.69897) + (0.84510) + 2$$
$$\log(3500) \approx 3.54407$$

- Use the rule for quotients, the rule for products, and the rule for $\log(10^x)$:

$$\log(5.5) = \log\left(\frac{55}{10}\right)$$
$$\log(5.5) = \log(55) - \log(10)$$
$$\log(5.5) = \log(5 \times 11) - \log(10)$$
$$\log(5.5) = \log(5) + \log(11) - \log(10)$$
$$\log(5.5) \approx (0.69897) + (1.04139) - 1$$
$$\log(5.5) \approx 0.74036$$

- Use both the rule for products and the rule for powers:

$$\log(77x^2) = \log\left(7 \times 11 \times x^2\right)$$
$$\log(77x^2) = \log(7) + \log(11) + \log(x^2)$$
$$\log(77x^2) = \log(7) + \log(11) + 2\log(x)$$
$$\log(77x^2) \approx (0.84510) + (1.04139) + 2\log(x)$$
$$\log(77x^2) \approx 1.88649 + 2\log(x)$$

- Use both the rule for quotients and the rule for powers:

$$\log\left(\frac{\sqrt{x}}{6.3}\right) = \log\left(\sqrt{x}\right) - \log(6.3)$$
$$\log\left(\frac{\sqrt{x}}{6.3}\right) = \log\left(x^{\frac{1}{2}}\right) - \log\left(\frac{63}{10}\right)$$
$$\log\left(\frac{\sqrt{x}}{6.3}\right) = \log\left(x^{\frac{1}{2}}\right) - \log\left(\frac{3^2 \times 7}{10}\right)$$
$$\log\left(\frac{\sqrt{x}}{6.3}\right) = \log\left(x^{\frac{1}{2}}\right) - \left[\log\left(3^2 \times 7\right) - \log(10)\right]$$

$$\log\left(\frac{\sqrt{x}}{6.3}\right) = \log\left(x^{\frac{1}{2}}\right) - \left[\log 3^2 + \log 7 - \log 10\right]$$

$$\log\left(\frac{\sqrt{x}}{6.3}\right) = \frac{1}{2}\log x - \left[2\log 3 + \log 7 - \log 10\right]$$

$$\log\left(\frac{\sqrt{x}}{6.3}\right) \approx \frac{1}{2}\log x - \left[2(0.47712) + (0.84510) - 1\right]$$

$$\log\left(\frac{\sqrt{x}}{6.3}\right) \approx \frac{1}{2}\log x - 0.79934$$

2. Use the various rules we have for logs of products, powers and quotients:

$$\ln\left[\frac{504x^2}{(y+1)^3}\right] = \ln(504) + \ln(x^2) - \ln(y+1)^3$$

$$\ln\left[\frac{504x^2}{(y+1)^3}\right] = \ln(504) + 2\ln(x) - 3\ln(y+1)$$

$$\ln\left[\frac{504x^2}{(y+1)^3}\right] = \ln(2^3 \times 7 \times 3^2) + 2\ln(x) - 3\ln(y+1)$$

$$\ln\left[\frac{504x^2}{(y+1)^3}\right] = 3\ln 2 + \ln 7 + 2\ln 3 + 2\ln(x) - 3\ln(y+1)$$

3. The question is, how to find the constants A and k in a "standard" model of exponential growth, $q(t) = \mathrm{A}e^{kt}$.

    - As we noted in the problem, the value of A came from weighing the substance at the initial time ($t = 0$). That's because $q(0) = \mathrm{A}e^{0t} = \mathrm{A}e^{0} = \mathrm{A}\cdot 1 = \mathrm{A}$.

    - Now we turn to figuring out the value of k. This is often done by plotting measured values of *q(t)* versus values of *t* on special graph paper so that the *logarithm* of the dependent variable is what gets plotted. The effect is to plot ln[*q(t)*] as the dependent variable, where *t* is the independent variable.

    - When we apply the natural log function to $q(t) = 400e^{kt}$, we get

$$\ln[q(t)] = \ln[400e^{kt}]$$

$$\ln[q(t)] = \ln 400 + \ln(e^{kt})$$

$$\ln[q(t)] = \ln 400 + kt\ln e$$

$$\ln[q(t)] = \ln 400 + kt$$

    - When we plot ln[*q(t)*] versus *t*, **we get a straight line.** Comparing the equation above to the slope-intercept form of a line ($y = \mathrm{m}x +$

b), we see that its vertical intercept is ln(400), and its *slope* is the desired coefficient k.

4. The question is, how to find the constants A and k in a "standard" model of exponential growth, $q(t) = Ae^{kt}$.

    - A typical model for exponential decay is

$$q(t) = 650e^{-0.07t}$$

    where *q(t)* is measured in grams and *t* in hours. We see that this model describes decay rather than growth because of the negative coefficient (–0.07) in the exponent.

    a. We know that when $t = 0$, $q(t) = q(0) = 650\ e^0 = 650$. Thus, at the beginning of the discussion, 650 grams of the material present were radioactive.

    b. After 3 hours (when $t = 3$), $q(t) = q(3) = 650\ e^{(-0.07)(3)} = 650\ e^{-0.21}$.

    To approximate $q(3)$ we need a calculator or a table of exponential values. Usually we do this by using a calculator: we find that $e^{-0.21} \cong 0.811$. *Notice that this value is less than 1! This is because the exponent's value is negative.*

    So, we find $q(3) \cong (650)(0.811) \cong 526.5$ grams.

    c. The substance's half-life is the time it takes for its radioactive portion to decrease by half. That would mean that the factor $e^{-0.07t}$ must be 0.5. Thus,

$$e^{-0.07t} = 0.5$$

$$\ln\left(e^{-0.07t}\right) = \ln(0.5)$$

$$-0.07t\ln(e) = \ln(0.5)$$

$$-0.07t \approx -0.693$$

$$t \approx \frac{-0.693}{-0.07}$$

$$t \approx 9.9$$

    The half-life of this substance is approximately 9.9 hours, which would be about 9 hours and 54 minutes.

5. We have put \$2500 in a bank account that pays 5 percent interest at the end of each year.

a. If we put \$2500 in the bank at 5% interest, at the end of one year we will have our original \$2500 plus (0.05)(2500) = \$125 in interest, or \$2625. As a percentage, we will have 105% of our original amount:

$$2500 + (0.05)(2500) = (2500)(1 + 0.05) = (2500)(1.05)$$

b. As long as the bank is still paying 5% interest, the balance at the *end* of any year will be 105% of the balance at the beginning of the year. Leaving N dollars in the account for 1 year means that the balance will be N(1.05). Now after each year, the balance will be multiplied by 1.05. Thus, after 10 years, the balance will be $N(1.05)^{10}$.

c. After 10 years, \$2500 at 5% interest will become

$$(2500)(1.05)^{10} \approx \$4,072.24$$

d. To rewrite $2500(1.05^t)$ in the "usual" exponential form

$$(2500)(1.05)^t = 2500e^{kt}$$

we would need to rewrite 1.05 as a power of e:

$$\begin{aligned}
(2500)(1.05)^t &= 2500e^{kt} \\
(1.05)^t &= e^{kt} \\
\ln(1.05)^t &= \ln e^{kt} \\
t\ln(1.05) &= kt\ln e \\
\ln(1.05) &= k\ln e \\
\ln(1.05) &= k \\
k &\approx 0.0488
\end{aligned}$$

Or, we might simply have remembered that $1.05 = e^{\ln(1.05)}$. That's what inverse functions do, after all: each undoes the effect of the other!

Thus,

$$\begin{aligned}
(1.05)^t &= e^{(\ln(1.05))t} \\
(1.05)^t &= e^{0.0488t}
\end{aligned}$$

e. Money in this account will triple when

$$\begin{aligned}
e^{0.0488t} &= 3 \\
\ln e^{0.0488t} &= \ln 3 \\
0.0488t\ln e &= \ln 3 \\
0.0488t &\approx 1.099 \\
t &\approx 22.52
\end{aligned}$$

So it will take 22 years, 6 months, and a few days for the money to triple.